

# Ordered glassy spin system

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A solid is typically deemed amorphous, or glassy, when there are no Bragg peaks in its diffraction pattern. We construct a two dimensional configuration of Ising spins with an autocorrelation function which vanishes at all nonzero distances, so that its scattering pattern is flat. This configuration is a ground state of a spin Hamiltonian with deterministic, translationally-invariant and local interactions. Despite ostensibly being amorphous, this configuration has perfect underlying order.

The question of whether a system is considered ordered is an evolving subject. Before 1984, order was equated with crystallinity; thereafter, quasicrystals—aperiodic solids with perfect long-range translational order—were shown to be possible[1]. All other solids were regarded as glassy, or amorphous. It is natural to ask whether there are any other types of organization which may be perfectly ordered, while being neither crystalline nor quasicrystalline [2, 3].

Recently, a new criterion for the definition and quantification of long-range spatial order was proposed[4, 5]. This definition subsumes crystals and quasicrystals, and includes many other systems which would have been classified as disordered due to their scattering spectrum. One example of this is the Rudin-Shapiro sequence[6], which may be thought of as a one-dimensional configuration of Ising spins, which has an absolutely continuous flat Fourier spectrum. Given that such configurations are mathematically possible, the next important question is whether they are physically relevant. That is: Can the configuration be the ground state of some reasonable Hamiltonian?

In this Letter we develop a Wang tiling[7] which is a two-dimensional generalization of the Rudin-Shapiro sequence. One interpretation of this tiling is as a configuration of Ising spins ( $\sigma = \pm 1$ ) on a square lattice. Although its autocorrelation function is identically zero except at the origin, it is perfectly ordered in the sense of Reference [4]. Moreover, we shall show that this tiling is the ground state of a short-range Hamiltonian. Although this system may be thought of as somewhat artificial, the fact that it minimizes the energy resulting from a local interaction lends credence to the suggestion that physical systems may realize ordered states which are neither crystalline nor quasicrystalline at low enough temperatures. Such states would not be distinguished by Bragg peaks in their scattering function, and thus might be incorrectly categorized as disordered.

In order to develop the tiling[8], we will first consider several related polynomials[9]. Denoting the spin at the (square) lattice vertex  $k, l$  by  $\sigma_{k,l}$ , we define the polynomial  $P(x, y)$  by

$$P(x, y) = \sum_{k,l} \sigma_{k,l} x^k y^l,$$

where  $x, y$  are complex variables. Note that if  $x, y$  are defined on the unit circle, we may write  $x = e^{iq_x}$  and  $y = e^{iq_y}$ , which, when substituted into  $P(x, y)$ , yields the lattice Fourier transform of the  $\{\sigma_{k,l}\}$ :

$$\mathcal{F}[\{\sigma\}] = \sum_{k,l} \sigma_{k,l} e^{ikq_x} e^{ilq_y}$$

Now, if the amplitude of the Fourier transform is a constant, independent of  $(q_x, q_y)$ , the autocorrelation function must vanish except at the origin:

$$\langle \sigma_{x,y} \sigma_{x+\xi, y+\eta} \rangle = \delta_{\xi,0} \delta_{\eta,0} \quad (1)$$

Thus the problem of finding a “deterministic amorphous” configuration of spins reduces to finding a deterministic polynomial in the variables  $x = e^{iq_x}$  and  $y = e^{iq_y}$  which has a constant modulus.

Let us now consider the polynomials  $P_n = P_n(x, y)$ , and define them recursively using three auxiliary polynomials,  $Q_n, R_n, S_n$ , where the dependence on  $x, y$  has been omitted for ease in reading:

$$\begin{aligned} P_{n+1} &= P_n + x^{2^n} Q_n + y^{2^n} R_n + (xy)^{2^n} S_n \\ Q_{n+1} &= P_n + x^{2^n} Q_n - y^{2^n} R_n - (xy)^{2^n} S_n \\ R_{n+1} &= P_n - x^{2^n} Q_n + y^{2^n} R_n - (xy)^{2^n} S_n \\ S_{n+1} &= P_n - x^{2^n} Q_n - y^{2^n} R_n + (xy)^{2^n} S_n \\ P_0 &= Q_0 = R_0 = S_0 = 1 \end{aligned} \quad (2)$$

$P_n, Q_n, R_n, S_n$  are all polynomials of degree  $2^{n+1} - 2$  and their coefficients take the values  $\pm 1$ . In particular,  $P_n(x, y) = \sum_{k,l=0}^{2^n-1} \sigma_{kl} x^k y^l$ , where  $\sigma_{kl} = \pm 1$ . Thus we may consider  $\sigma_{kl}$  to be a configuration of Ising spins on a square lattice of linear size  $2^n$ , with the total number of spins being  $N = 4^n$ .

Assuming now that  $x$  and  $y$  have unit modulus, it is easily verified that

$$\begin{aligned} |P_{n+1} + Q_{n+1} + R_{n+1} + S_{n+1}|^2 &= 16|P_n|^2 \\ |P_{n+1} + Q_{n+1} - R_{n+1} - S_{n+1}|^2 &= 16|Q_n|^2 \\ |P_{n+1} - Q_{n+1} + R_{n+1} - S_{n+1}|^2 &= 16|R_n|^2 \\ |P_{n+1} - Q_{n+1} - R_{n+1} + S_{n+1}|^2 &= 16|S_n|^2 \end{aligned}$$

Summing these, we get

$$\begin{aligned} |P_{n+1}|^2 + |Q_{n+1}|^2 + |R_{n+1}|^2 + |S_{n+1}|^2 \\ = 4(|P_n|^2 + |Q_n|^2 + |R_n|^2 + |S_n|^2) \end{aligned}$$

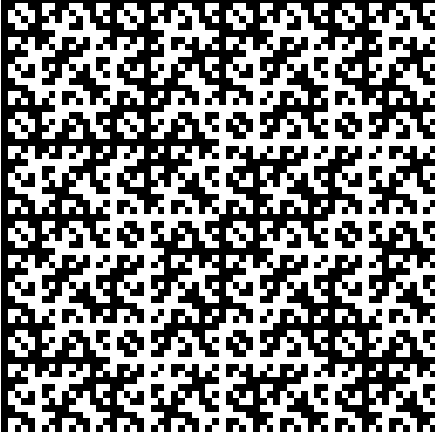


FIG. 1. Part of the configuration of Ising spins that is a generalization of the Rudin-Shapiro sequence. Each black square corresponds to a spin  $\sigma = 1$ , and each white square corresponds to a spin  $\sigma = -1$ . Here  $0 \leq x, y < 64$ .

with  $|P_0|^2 + |Q_0|^2 + |R_0|^2 + |S_0|^2 = 4$ . This is a simple recursion whose solution is

$$|P_n|^2 + |Q_n|^2 + |R_n|^2 + |S_n|^2 = 4^{n+1}. \quad (3)$$

This immediately implies that  $|P_n| \leq 2^{n+1}$ , which scales as the square root of the number of spins; this implies an absolutely continuous Fourier spectrum[10]. It is, in fact, possible to say more: in the thermodynamic limit the Fourier spectrum is *flat*, i.e.  $|P_n|$  is a constant independent of  $(q_x, q_y)$ [11], and

$$|P_n| \rightarrow \sqrt{N}, \quad (\text{as } N \rightarrow \infty) \quad (4)$$

The terms in  $P_n$ , which are all of the form  $\pm x^k y^l$ , can be organized in the form of a  $2^n \times 2^n$  matrix  $\mathcal{P}_n$  such that if  $\sigma_{k,l} x^k y^l \in P_n(x, y)$  then  $(\mathcal{P}_n)_{l,k} = \sigma_{k,l}$ . In this manner we also define matrices for the three auxilliary polynomials, and denote them  $\mathcal{Q}_n, \mathcal{R}_n, \mathcal{S}_n$ . This allows us to write the recursive definition (2) as

$$\begin{aligned} \mathcal{P}_{n+1} &= \begin{pmatrix} \mathcal{P}_n & \mathcal{Q}_n \\ \mathcal{R}_n & \mathcal{S}_n \end{pmatrix} & \mathcal{Q}_{n+1} &= \begin{pmatrix} \mathcal{P}_n & \mathcal{Q}_n \\ -\mathcal{R}_n & -\mathcal{S}_n \end{pmatrix} \\ \mathcal{R}_{n+1} &= \begin{pmatrix} \mathcal{P}_n & -\mathcal{Q}_n \\ \mathcal{R}_n & -\mathcal{S}_n \end{pmatrix} & \mathcal{S}_{n+1} &= \begin{pmatrix} \mathcal{P}_n & -\mathcal{Q}_n \\ -\mathcal{R}_n & \mathcal{S}_n \end{pmatrix} \end{aligned} \quad (5)$$

with  $\mathcal{P}_0 = \mathcal{Q}_0 = \mathcal{R}_0 = \mathcal{S}_0 = 1$ .

In each iteration, the linear size of the matrices  $\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n, \mathcal{S}_n$  is doubled. After  $n$  iterations each matrix has  $N = 4^n$  entries, which are  $\pm 1$ . We regard the matrix  $\mathcal{P}_n$  as a configuration of Ising spins on a square lattice; a configuration of  $64^2$  spins ( $n = 6$ ) is shown in Fig. 1.

*Substitution Rules.* Eq. (5) implies a recursive construction rule with which we may construct a Wang tiling. By looking at the matrix  $\mathcal{P}_{n+1}$ , we note that a configuration at generation  $n$  can be subdivided into four



FIG. 2. (color online) The configuration of spins,  $\mathcal{P}$ , in terms of the eight letters  $\alpha_+, \alpha_-, \beta_+, \beta_-, \gamma_+, \gamma_-, \delta_+, \delta_-$  for  $0 \leq x, y < 256$ . Since each four spins are assigned one letter there are only  $128 \times 128$  pixels in the figure.

quadrants; each representing a configuration at generation  $n - 1$ . These quadrants may again be subdivided. This iterative procedure “bottoms out” at generation, say[12],  $n = 1$ . The configuration  $\mathcal{P}_n$  is now subdivided into  $2 \times 2$  blocks of the form  $\pm \mathcal{P}_1, \pm \mathcal{Q}_1, \pm \mathcal{R}_1, \pm \mathcal{S}_1$ . If we simultaneously replace these  $2 \times 2$  blocks by  $4 \times 4$  blocks as prescribed by Eq. (5) we transform  $\mathcal{P}_n$  into  $\mathcal{P}_{n+1}$ . This procedure allows us to forgo configurations  $\pm \mathcal{Q}_n, \pm \mathcal{R}_n, \pm \mathcal{S}_n$  for  $n > 1$  and inflate the configuration  $\mathcal{P}_n \rightarrow \mathcal{P}_{n+1}$  directly.

Let us now denote each of the eight  $2 \times 2$  blocks by a letter from the alphabet  $\{\alpha_+, \alpha_-, \beta_+, \beta_-, \gamma_+, \gamma_-, \delta_+, \delta_-\}$  as follows

$$\begin{aligned} \alpha_+ &= -\alpha_- = \mathcal{P}_1 = \begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix}, \\ \beta_+ &= -\beta_- = \mathcal{Q}_1 = \begin{pmatrix} +1 & +1 \\ -1 & -1 \end{pmatrix}, \\ \gamma_+ &= -\gamma_- = \mathcal{R}_1 = \begin{pmatrix} +1 & -1 \\ +1 & -1 \end{pmatrix}, \\ \delta_+ &= -\delta_- = \mathcal{S}_1 = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}, \end{aligned} \quad (6)$$

Each of these letters will represent a set of tiles in the Wang tiling. In terms of these letters, the configuration  $\mathcal{P}_n$  is shown for  $n = 8$  in Figure 2. For ease in visualization, we will represent each of the tiles as a  $2 \times 2$  ‘semaphore’ block, with 1 represented by a black square and -1 by a white square, as shown in Figure 3.

By Eq. (5), each of the above  $2 \times 2$  blocks is to be

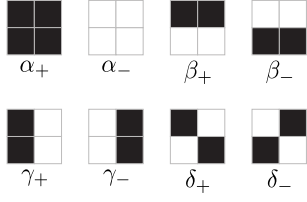


FIG. 3. The eight basic tiles in the ‘semaphore’ representation, in which a black square represents a spin  $\sigma = 1$ , derived from Eq. (6).

substituted by a  $4 \times 4$  block in the following manner

$$\begin{aligned}
 \alpha_+ &\rightarrow \begin{array}{cc} \alpha_+ & \beta_+ \\ \gamma_+ & \delta_+ \end{array} & \alpha_- &\rightarrow \begin{array}{cc} \alpha_- & \beta_- \\ \gamma_- & \delta_- \end{array} \\
 \beta_+ &\rightarrow \begin{array}{cc} \alpha_+ & \beta_+ \\ \gamma_- & \delta_- \end{array} & \beta_- &\rightarrow \begin{array}{cc} \alpha_- & \beta_- \\ \gamma_+ & \delta_+ \end{array} \\
 \gamma_+ &\rightarrow \begin{array}{cc} \alpha_+ & \beta_- \\ \gamma_+ & \delta_- \end{array} & \gamma_- &\rightarrow \begin{array}{cc} \alpha_- & \beta_+ \\ \gamma_- & \delta_+ \end{array} \\
 \delta_+ &\rightarrow \begin{array}{cc} \alpha_+ & \beta_- \\ \gamma_- & \delta_+ \end{array} & \delta_- &\rightarrow \begin{array}{cc} \alpha_- & \beta_+ \\ \gamma_+ & \delta_- \end{array}.
 \end{aligned} \tag{7}$$

These substitution rules are shown in Figure 4. The substitution rules obey two important properties: (i) each of the pairs  $(\alpha_+, \alpha_-)$ ,  $(\beta_+, \beta_-)$ ,  $(\gamma_+, \gamma_-)$ , and  $(\delta_+, \delta_-)$  occupy a specific quadrant on the rhs of each substitution rule, (ii) in each pair, one partner is the negative of the other, and its substitution rule is the negative of its partner’s substitution rule. By repeatedly applying these rules, we can generate a tiling of arbitrary size. The amplitude of the Fourier transform of configurations arising from substitution rules obeying these two properties was rigorously shown[11] to be flat, independent of wave vector.

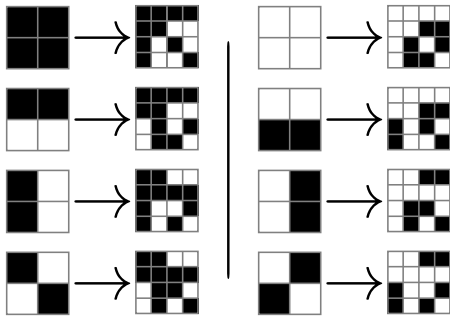


FIG. 4. The two-dimensional substitution rules, corresponding to Eq. (7). These substitutions generate the configuration  $\mathcal{P}$  when iterated on a  $2 \times 2$  block of all black squares.

**Tiling model.** By a theorem due to Mozes [13], configurations emerging from substitution rules such as those in Eq. (7) may be enforced by a set of Wang tiles—unit squares with marked edges. The plane is to be tiled by copies of these tiles [14] such that two tiles may abut only

if the markings on their shared edge match—these are the so-called *matching rules*. We note that although the configuration above is generated substitutionally from eight basic tiles, there will be more than eight Wang tiles in terms of their markings.

By associating an energy penalty  $+J$  for each edge with mismatched markings, we define a Hamiltonian

$$\mathcal{H} = J \sum_{x,y} 2 - \delta(T_{x,y}^N - T_{x,y+1}^S) - \delta(T_{x,y}^E - T_{x+1,y}^W) \tag{8}$$

where  $\delta(0) = 1$  and where  $T$  is a tile with edge markings  $T^N, T^S, T^E, T^W$ . The ground state of this Hamiltonian has zero energy and corresponds to a tiling in which there are no mismatches.

We note that this system may be regarded as a generalized Potts model. Let  $t$  be a Potts spin, taking  $M$  possible values, where  $M$  is the number of distinct Wang tiles (by Mozes’ theorem,  $M$  is finite). We denote these values by  $t = 1 \dots M$ . Since two nearest-neighbor tiles have an interaction energy ( $+J$  or  $0$ ), the Hamiltonian (8) can be written as

$$\mathcal{H} = J \sum_{x,y} Q^{NS}(t_{x,y}, t_{x,y+1}) + Q^{EW}(t_{x,y}, t_{x+1,y}) \tag{9}$$

where  $Q^{NS}$  and  $Q^{EW}$  are two  $M \times M$  matrices whose entries are 0 and 1. In a regular Potts model each  $Q$  matrix would have zeros on the diagonal and ones off the diagonal; here the matrices reflect the matching rules between pairs of tiles in their various juxtapositions.

In order to fully define the system, we must construct the matrices  $Q$  used in Eq. (9), or equivalently, construct a set of Wang tiles which allow our configuration as the only legal tiling of the plane. We have constructed such a set by the two-step procedure described below.

**Step 1.** Apply Mozes’ construction [13] to our tiling. In his construction there are two fundamental types of tiles. The first type corresponds to the substitutional (“letter”) tiles (in our case  $\alpha_+, \alpha_-, \beta_+, \beta_-, \gamma_+, \gamma_-, \delta_+, \delta_-$ ). These tiles are constrained to occupy the sites of a diluted lattice of double the spacing of the physical lattice (*e.g.*, sites where  $x$  and  $y$  are both even). The second type corresponds to tiles whose function is to synchronize [13] the letter tiles so that those will order into the tiling shown in Fig. 2. The “synchronizer” tiles are not allowed to occupy the sites of the diluted lattice. This construction leads a tile set consisting of 116 tiles [15], of which 8 are letter tiles and 108 are synchronizer tiles. The Hamiltonian is not yet that of Eq. (8) because different tiles are restricted to different sites of the lattice.

**Step 2.** Define “supertiles”, each a plaquette consisting of one letter tile and the three synchronizer tiles lying to its right, bottom, and bottom-right. The edge-markings of the supertiles are the superposition of the

edge-markings of its constituent tiles. A careful combinatorial counting shows that this step generates 400 supertiles, for which the Hamiltonian is that of Eq. (8).

The 400 tiles differ by their edge markings but represent variations of the letter tiles  $\{\alpha_+, \alpha_-, \beta_+, \beta_-, \gamma_+, \gamma_-, \delta_+, \delta_-\}$ . By (6), the tiles  $\{\alpha_+, \beta_+, \gamma_+, \delta_+\}$  are the negatives of the tiles  $\{\alpha_-, \beta_-, \gamma_-, \delta_-\}$ , and indeed if we color the first set black and the second set white, the ground state of the Hamiltonian (8) is the configuration shown in Fig. 1.

*Underlying Order.* Although its two-point function vanishes (1), our tiling is deterministic, and has subextensive patch entropy[4]. Thus, we expect there to be some measure which would differentiate between it and a random arrangement of the same tiles. It is important to note that there is no *a priori* way of knowing what measure would reveal the order of an arbitrary given tiling, but, for the system we examine in this paper, we have identified one such four-point correlation function

$$\pi(k, l) = \sigma_{k,l} \sigma_{k+1,l} \sigma_{k,l+1} \sigma_{k+1,l+1}$$

which is useful in revealing the underlying hierarchical order of our tiling.

This is essentially a parity function as it returns +1 when there are an even number of  $\sigma = -1$  spins in the quadruplet of spins in which  $k, l$  is the upper left position, and  $-1$  when there are an odd number of  $\sigma = -1$  spins. This function is plotted in Fig. 5. The average of this four-point correlation function is  $\langle \pi \rangle = (1/N) \sum_{k,l} \pi(k, l)$ , and by direct inspection of Fig. 5, we see that one third of the sites of the the ground state have  $\pi = 1$  and the other two thirds have  $\pi = -1$ . Therefore

$$\langle \pi \rangle = -\frac{1}{3}$$

and the four-point correlations, unlike the two-point correlation, are not generically zero. As may be seen from Fig. 5, the function  $\pi(k, l)$  may be regarded as a superposition of lattices. The first lattice has a two-site spacing between sites with  $\pi = 1$ . The second lattice has a four-site spacing between sites with  $\pi = 1$ , etc. This is a superposition of lattices in which the level  $n$  lattice ( $n \geq 1$ ) has a lattice constant of  $2^n$ .

We have constructed a deterministic two-dimensional system of Ising spins whose ground state is glassy in the sense that its autocorrelation function vanishes exactly in the thermodynamic limit. This configuration may be generated by a recursive method, which guarantees that its patch entropy [4] scales logarithmically with patch size. In this sense, the ground state configuration is ordered. A tiling which enforces a hierarchical structure, albeit on a triangular lattice, has recently been studied by Byington and Socolar [16] and shown to exhibit an infinite sequence of phase transitions; The finite temperature properties of our system should also prove intriguing, and will be detailed in a future publication.

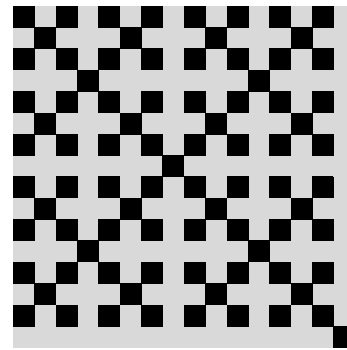


FIG. 5. The parity function  $\pi(k, l)$  for  $0 \leq k, l < 16$ . Black squares represent  $\pi(k, l) = 1$  and light gray squares represent  $\pi(k, l) = -1$ .

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